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INTERTWINING CONNECTIVITY IN MATROIDS

RONG CHEN AND GEOFF WHITTLE

ABSTRACT. Let M be a matroid and let Q, R, S and T be subsets of the ground set such that the smallest separation that separates Q from R has order k and the smallest separation that separates S from T has order ℓ . We prove that if $E(M) - (Q \cup R \cup S \cup T)$ is sufficiently large, then there is an element e of M such that, in one of $M \setminus e$ or M/e , both connectivities are preserved.

1. INTRODUCTION

Let M be a matroid with ground set $E(M)$. For any $X \subseteq E(M)$, define $\lambda_M(X) := r_M(X) + r_M(E(M) - X) - r(M)$. For disjoint subsets Q, R of $E(M)$, the *connectivity between Q and R* is

$$\kappa_M(Q, R) := \min\{\lambda_M(X) : Q \subseteq X \subseteq E(M) - R\}.$$

In the paper, we prove

Theorem 1.1. *There is a function $c : \mathbb{N}^2 \rightarrow \mathbb{N}$ with the following property. Let M be a matroid, and $Q, R, S, T, F \subseteq E(M)$ sets of elements such that $Q \cap R = S \cap T = \emptyset$ and $F = E(M) - (Q \cup R \cup S \cup T)$. Let $k := \kappa_M(Q, R)$ and $\ell := \kappa_M(S, T)$. If $|F| \geq c(k, \ell)$, then there is an element $e \in F$ such that one of the following holds:*

- (i) $\kappa_{M \setminus e}(Q, R) = k$ and $\kappa_{M \setminus e}(S, T) = \ell$;
- (ii) $\kappa_{M/e}(Q, R) = k$ and $\kappa_{M/e}(S, T) = \ell$.

This theorem resolves a conjecture of Geelen (private communication). It strengthens a theorem of Huynh and van Zwam [2] who prove the result for a class that includes all representable matroids but does not include all matroids.

The value that we give for $c(k, \ell)$ is unlikely to be tight. The $(k+1) \times (\ell+1)$ grid gives an example where the theorem fails with $|F| = 2kl - l - k$. Perhaps this example is extremal?

Conjecture 1.2. *Theorem 1.1 holds with $|F| = 2kl - l - k + 1$.*

2. PROOF OF THEOREM 1.1

For any disjoint subsets Q, R of the ground set of a matroid M , Tutte [3] proved that there is a minor N of M with $E(N) = Q \cup R$ and such that $\kappa(Q, R) = \lambda_N(Q)$, which is a generalization of Menger's theorem to matroids. Equivalently, we have

Lemma 2.1. *Let M be a matroid and Q, R be disjoint subsets of $E(M)$. For any $e \in E(M) - (Q \cup R)$ either $\kappa_{M \setminus e}(Q, R) = \kappa_M(Q, R)$ or $\kappa_{M/e}(Q, R) = \kappa_M(Q, R)$.*

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Let M be a matroid and Q, R be disjoint subsets of $E(M)$. Define $\square_M(Q, R) := r_M(Q) + r_M(R) - r_M(Q \cup R)$. A partition (A, B) of $E(M)$ is $Q - R$ -separating of order $k + 1$ if $Q \subseteq A$, $R \subseteq B$ and $\lambda_M(A) \leq k$. Let $e \in E(M) - (Q \cup R)$. If $\kappa_{M \setminus e}(Q, R) = \kappa_M(Q, R)$, then e is deletable with respect to (Q, R) ; if $\kappa_{M/e}(Q, R) = \kappa_M(Q, R)$, then e is contractible with respect to (Q, R) ; and if e is both deletable and contractible with respect to (Q, R) , then e is flexible with respect to (Q, R) . Lemma 2.1 implies that for any $e \in E(M) - (Q \cup R)$ either e is deletable with respect to (Q, R) or e is contractible with respect to (Q, R) .

Theorem 2.2. ([2], Theorem 3.4.) *Let M be a matroid and Q, R be disjoint subsets of $E(M)$, let $k := \kappa(Q, R)$, and let $F \subseteq E(M) - (Q \cup R)$ be a set of non-flexible elements. There are an ordering (f_1, \dots, f_n) of F and a sequence of (A_1, \dots, A_n) of subsets of $E(M)$ such that*

- (i) A_i is $Q - R$ -separating of order $k + 1$ for each $i \in \{1, \dots, n\}$;
- (ii) $A_i \subseteq A_{i+1}$ for each $i \in \{1, \dots, n\}$;
- (iii) $A_i \cap F = \{f_1, \dots, f_i\}$ for each $i \in \{1, \dots, n\}$;
- (iv) $f_i \in \text{cl}(A_i - \{f_i\}) \cap \text{cl}(E(M) - A_i)$ or $f_i \in \text{cl}^*(A_i - \{f_i\}) \cap \text{cl}^*(E(M) - A_i)$.

Theorem 2.3. ([2], Lemma 3.6.) *Let M be a matroid and Q, R be disjoint subsets of $E(M)$, let $k := \kappa(Q, R)$, and let $(U, E(M) - U)$ be a $Q - R$ -separating set of order $k + 1$. If $e \in E(M) - (U \cup R)$ is non-contradictable with respect to (Q, R) , then e is also non-contradictable with respect to (U, R) .*

First we prove that Theorem 1.1 holds for the case $|S| = |T| = \ell$.

Lemma 2.4. *There is a function $c : \mathbb{N}^2 \rightarrow \mathbb{N}$ with the following property. Let M be a matroid, and $Q, R, S, T, F \subseteq E(M)$ sets of elements such that $Q \cap R = S \cap T = \emptyset$ and $F = E(M) - (Q \cup R \cup S \cup T)$. Let $k := \kappa_M(Q, R)$ and $\ell := \kappa_M(S, T)$. If $|S| = |T| = \ell$ and $|F| \geq c(k, \ell)$, then there is an element $e \in F$ such that one of the following holds:*

- (i) $\kappa_{M \setminus e}(Q, R) = k$ and $\kappa_{M \setminus e}(S, T) = \ell$;
- (ii) $\kappa_{M/e}(Q, R) = k$ and $\kappa_{M/e}(S, T) = \ell$.

Proof. We prove that the result holds for $c(k, \ell) := (2\ell + 1)2^{2k+1}$. If F contains some flexible element with respect to (Q, R) or (S, T) , then we are done. So we may assume that each element in F is non-flexible with respect to (Q, R) and non-flexible with respect to (S, T) . By Lemma 2.1 an element e in F is deletable (or contractible) with respect to (Q, R) if and only if e is contractible (or deletable) with respect to (S, T) , for otherwise the lemma holds.

Let $(A_1, \dots, A_{c(k, \ell)})$ be the nested sequence of $Q - R$ separating sets from Theorem 2.2, let $(B_1, \dots, B_{c(k, \ell)})$ be their complements, and let $(f_1, \dots, f_{c(k, \ell)})$ be the corresponding ordering of F . Since $|S| = |T| = \ell$, there is a positive integer i such that $i + 2^{2k+1} \leq c(k, \ell)$ and such that $Q \cup R \cup S \cup T \subseteq A_i \cup B_{i+2^{2k+1}}$. Set

$$\begin{aligned} Q' &:= A_i, \quad R' := B_{i+2^{2k+1}}, \quad F' := E(M) - (Q' \cup R'), \\ A'_j &:= A_{i+j}, \quad B'_j := B_{i+j}, \quad f'_j := f_{i+j}, \quad \text{for any } 1 \leq j \leq 2^{2k+1}. \end{aligned}$$

That is, $F' = \{f'_1, \dots, f'_{2^{2k+1}}\}$. By duality and Lemma 2.3, each element in F' is non-flexible with respect to (Q', R') .

Let $(C_1, \dots, C_{2^{2k+1}})$ be the nested sequence of $S - T$ separating sets from Theorem 2.2 determined by the non-flexible-element set F' with respect to (S, T) , let $(D_1, \dots, D_{2^{2k+1}})$ be their complements, and let $(g_1, \dots, g_{2^{2k+1}})$ be the corresponding ordering of F' . By

duality we may assume that g_1 is a deletable element with respect to (S, T) . Then (i) $g_1 \in \text{cl}(C_1 - \{g_1\})$ and (ii) g_1 is a contractible element with respect to (Q, R) . By (i) and the fact that $C_1 - \{g_1\} \subseteq Q' \cup R'$ we see that $g_1 \in \text{cl}(Q' \cup R')$. From (ii) we deduce that $g_1 \notin \text{cl}(Q')$ and $g_1 \notin \text{cl}(R')$. Therefore $\square_M(Q' \cup \{g_1\}, R') = \square_M(Q', R') + 1$. Assume that $g_1 = f'_j$. If $j \leq 2^{2k}$ then set $Q'' := A'_j, R'' := R'$; else if $j > 2^{2k}$ then set $Q'' := Q', R'' := B'_{j-1}$. No matter which case happens, set $F'' := E(M) - (Q'' \cup R'')$. Evidently, $|F''| \geq 2^{2k}$ as $|F'| = 2^{2k+1}$. Replacing Q', R', F' with Q'', R'', F'' respectively and repeating the above analysis $2k$ times, there are numbers j_1, j_2 with $2k + 1 \leq j_1 \leq j_2 \leq 2^{2k+1}$ such that $\square_M(A'_{j_1}, B'_{j_2}) \geq k + 1$ or $\square_{M^*}(A'_{j_1}, B'_{j_2}) \geq k + 1$, a contradiction to the fact that $\lambda(A'_{j_1}) = k$. So the lemma holds. \square

To prove Theorem 1.1 we still need the following lemma.

Lemma 2.5. ([1], Lemma 4.7.) *Let M be a matroid and S, T be disjoint subsets of $E(M)$. There exists sets $S_1 \subseteq S, T_1 \subseteq T$ such that $|S_1| = |T_1| = \kappa(S_1, T_1)$.*

For convenience we restate Theorem 1.1 here.

Theorem 2.6. *There is a function $c : \mathbb{N}^2 \rightarrow \mathbb{N}$ with the following property. Let M be a matroid, and $Q, R, S, T, F \subseteq E(M)$ sets of elements such that $Q \cap R = S \cap T = \emptyset$ and $F = E(M) - (Q \cup R \cup S \cup T)$. Let $k := \kappa_M(Q, R)$ and $\ell := \kappa(S, T)$. If $|F| \geq c(k, \ell)$, then there is an element $e \in F$ such that one of the following holds:*

- (i) $\kappa_{M \setminus e}(Q, R) = k$ and $\kappa_{M \setminus e}(S, T) = \ell$;
- (ii) $\kappa_{M/e}(Q, R) = k$ and $\kappa_{M/e}(S, T) = \ell$.

Proof. We prove that the result holds for $c(k, \ell) := (2\ell + 1)2^{2k+1}$. By Lemma 2.5 there are sets $S_1 \subseteq S, T_1 \subseteq T$ such that $|S_1| = |T_1| = \kappa_M(S_1, T_1)$. Then Lemma 2.4 implies that there is an element $e_1 \in E(M) - (Q \cup R \cup S_1 \cup T_1)$ such that for some $M_1 \in \{M \setminus e_1, M/e_1\}$ we have $\kappa_{M_1}(Q, R) = k$ and $\kappa_{M_1}(S_1, T_1) = \ell$. Since $\kappa_{M_1}(S_1, T_1) = \ell$ implies $\kappa_{M_1}(S, T) = \ell$, when $e_1 \in F$ the lemma holds. So we may assume that $e_1 \notin F$. That is, $e_1 \in (S \cup T) - (S_1 \cup T_1)$. Since $F \subseteq E(M_1) - (Q \cup R \cup S_1 \cup T_1)$, using Lemma 2.4 again there is an element $e_2 \in E(M_1) - (Q \cup R \cup S_1 \cup T_1)$ such that for some $M_2 \in \{M_1 \setminus e_2, M_1/e_2\}$ we have $\kappa_{M_2}(Q, R) = k$ and $\kappa_{M_2}(S_1, T_1) = \ell$. Without loss of generality we may assume that $M_2 = M_1 \setminus e_2$. Then $\kappa_{M \setminus e_2}(Q, R) = k$ and $\kappa_{M \setminus e_2}(S_1, T_1) = \ell$ as $\kappa_M(Q, R) = k$ and $\kappa_M(S_1, T_1) = \ell$. Thus, when $e_2 \in F$, the lemma holds. So we may assume that $e_2 \notin F$. Since $(S \cup T) - (S_1 \cup T_1)$ is finite, repeating the above analysis several times we can always find a minor with an element e such that (i) or (ii) holds. The theorem follows from this observation and the fact that the connectivity function is monotone under minors. \square

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CENTER FOR DISCRETE MATHEMATICS, FUZHOU UNIVERSITY, FUZHOU, P. R. CHINA

SCHOOL OF MATHEMATICS, STATISTICS AND OPERATIONS RESEARCH, VICTORIA UNIVERSITY OF
WELLINGTON, NEW ZEALAND